THE EQUIVALENCE BETWEEN A NEW MULTISTEP ITERATION, S-ITERATION AND SOME OTHER ITERATIVE SCHEMES

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ABSTRACT. In this paper, we show that Picard, Krasnoselskij, Mann, Ishikawa, new two step, Noor, multistep, new multistep, SP and S-iterative schemes are equivalent for contractive-like operators.

1. Introduction and Preliminaries

In the last four decades, attention of researchers has been focused on the introduction and the convergences of various iteration procedures, e.g. see [7, 14, 18, 21, 30, 26, 16, 11], among others, for approximate fixed points of certain classes of self-operators. During the past 11 years, a large literature has developed around theme establishing equivalence between convergences of some well-known iterative schemes deal with various classes of operators. The authors who have made contributions to the study of equivalence between various iterative schemes are Rhoades and Şoltuz [[1]-[8]], Berinde [28], Şoltuz [22, 23], Olaleru and Akewe [13], Chang et al [20] and several references therein.

The aim of this paper is to show equivalence between convergences of a new multistep iteration, which is unifies and developes the iterative algorithms presented in [26] and [16], S-iteration and some other iterative schemes.

As a background of our exposition, we now mention some contractive mappings and iteration schemes.

In [27] Zamfirescu established an important generalization of the Banach fixed point theorem using the following contractive condition: For a mapping $T : E \to E$, there exist real numbers a, b, c satisfying $0 < a < 1$, $0 < b, c < 1/2$ such that, for each pair $x, y \in X$, at least one of the following is true:

$$\left\{ \begin{array}{ll} (\mathrm{z}_1) & \|Tx - Ty\| \leq a \|x - y\|, \\ (\mathrm{z}_2) & \|Tx - Ty\| \leq b (\|x - Tx\| + \|y - Ty\|), \\ (\mathrm{z}_3) & \|Tx - Ty\| \leq c (\|x - Ty\| + \|y - Tx\|). \end{array} \right. \tag{1.1}$$

A mapping T satisfying the contractive conditions (z_1), (z_2) and (z_3) in (1.1) is called a Zamfirescu operator. An operator satisfying condition (z_2) is called a *Kannan operator*, while the mapping satisfying condition (z_3) is called a *Chatterjea operator*. As shown in [29], the contractive condition (1.1) leads to

$$\left\{ \begin{array}{lll} (\mathrm{b}_1) & \|Tx - Ty\| \leq \delta \|x - y\| + 2\delta \|x - Tx\| & \text{if one use } (\mathrm{z}_2), \\ \text{and} & & \\ (\mathrm{b}_2) & \|Tx - Ty\| \leq \delta \|x - y\| + 2\delta \|x - Ty\| & \text{if one use } (\mathrm{z}_3), \end{array} \right. \tag{1.2}$$

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for all $x, y \in E$ where $\delta := \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}$, $\delta \in [0,1)$, and it was shown that this class of operators is wider than the class of Zamfirescu operators. Any mapping satisfying condition (b_1) or (b_2) is called a quasi-contractive operator.

Extending the above definition, Osilike and Udomene [15] considered operators T for which there exist real numbers $L \geq 0$ and $\delta \in [0,1)$ such that for all x, $y \in E$,

$$\|Tx - Ty\| \leq \delta \|x - y\| + L \|x - Tx\| . \tag{1.3}$$

Imoru and Olantiwo [9] gave a more general definition: The operator T is called a contractive-like operator if there exists a constant $\delta \in [0,1)$ and a strictly increasing and continuous function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$, such that, for each $x, y \in E$,

$$\|Tx - Ty\| \leq \delta \|x - y\| + \varphi(\|x - Tx\|) . \tag{1.4}$$

A map satisfying (1.4) need not have a fixed point, even if E is complete. For example, let $E = [0, \infty)$, and define T by

$$Tx = \begin{cases} 1.0, & 0 \leq x \leq 0.8, \\ 0.6, & 0.8 < x. \end{cases}$$

Without loss of generality we may assume that $x < y$. Then, for $0 \leq x < y \leq 0.8$ or $0.8 < x < y$, $\|Tx - Ty\| = 0$, and (1.4) automatically satisfied.

If $0 \leq x \leq 0.8 < y$, then $\|Tx - Ty\| = 0.4$.

Define φ by $\varphi(t) = Lt$ for any $L \geq 2$. Then φ is increasing, continuous, and $\varphi(0) = 0$. Also, $\|x - Tx\| = 1 - x$, so that $\varphi(\|x - Tx\|) = L(1 - x) \geq 0.2L \geq 0.4$.

Therefore

$$0.4 = \|Tx - Ty\| \leq L \|x - Tx\| \leq \delta \|x - y\| + L \|x - Tx\|$$

for any $0 \leq \delta < 1$, and (1.4) is satisfied for $0 \leq x \leq 0.8 < y$. But T has no fixed point.

However, using (1.4) it is obvious that, if T has a fixed point, then it is unique.

Throughout this paper $\mathbb{N}$ denotes the set of all nonnegative integers. Let X be a Banach space and $E \subset X$ a nonempty closed, convex subset of X, and T a self map on E. Define $F_T := \{p \in X : \ p = Tp\}$ to be the set of fixed points of T. Let $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$,$\{\gamma_n\}_{n=0}^{\infty}$ and $\left\{\beta_n^i\right\}_{n=0}^{\infty}$, $i = \overline{1, k-2}$, $k \geq 2$ be real sequences in $[0,1)$ satisfying certain conditions.

It is well known that Picard iteration procedure is defined by

$$\begin{cases} x_0 \in E, \\ x_{n+1} = Tx_n, \ \ n \in \mathbb{N}. \end{cases} \tag{1.5}$$

The Mann iterative scheme [30] is defined by

$$\begin{cases} x_0 \in E, \\ x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T x_n, \ \ n \in \mathbb{N}. \end{cases} \tag{1.6}$$

Taking $\alpha_n = \lambda$ (constant) in (1.6), we get Krasnoselskij iteration procedure as follows

$$\begin{cases} x_0 \in E, \\ x_{n+1} = (1 - \lambda) x_n + \lambda T x_n, \ \ n \in \mathbb{N}. \end{cases} \tag{1.7}$$

A sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\left\{\begin{array}{r l} x_0 \in E, & \\ y_n = (1-\beta_n)\,x_n + \beta_n T x_n, & \\ x_{n+1} = (1-\alpha_n)\,x_n + \alpha_n T y_n, & n \in \mathbb{N}, \end{array}\right. \tag{1.8}$$

is commonly known as the Ishikawa iterative method [21].

The following iteration scheme introduced by Noor [14]

$$\left\{\begin{array}{r l} x_0 \in E, & \\ z_n = (1-\gamma_n)\,x_n + \gamma_n T x_n, & \\ y_n = (1-\beta_n)\,x_n + \beta_n T z_n, & \\ x_{n+1} = (1-\alpha_n)\,x_n + \alpha_n T y_n, & n \in \mathbb{N}. \end{array}\right. \tag{1.9}$$

In year 2004, Rhoades and Şoltuz introduced in [7] a multistep procedure defined by

$$\left\{\begin{array}{l} x_0 \in E, \\ y_n^{k-1} = \left(1-\beta_n^{k-1}\right) x_n + \beta_n^{k-1} T x_n,\ k \geq 2, \\ y_n^{i} = \left(1-\beta_n^{i}\right) x_n + \beta_n^{i} T y_n^{i+1},\ i = \overline{1, k-2}, \\ x_{n+1} = (1-\alpha_n)\,x_n + \alpha_n T y_n^1,\ \ n \in \mathbb{N}. \end{array}\right. \tag{1.10}$$

The iteration processes (1.5), (1.6), (1.7), (1.8) and (1.9) can be viewed as the special cases of the iteration procedure (1.10).

The sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\left\{\begin{array}{l} x_0 \in E, \\ x_{n+1} = (1-\alpha_n)\,T x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)\,x_n + \beta_n T x_n,\ n \in \mathbb{N} \end{array}\right. \tag{1.11}$$

is known as the S-iteration process [10, 17, 18].

In 2008, S.Thianwan [26] defined the following iterative process

$$\left\{\begin{array}{l} x_0 \in E, \\ x_{n+1} = (1-\alpha_n)\,y_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)\,x_n + \beta_n T x_n,\ n \in \mathbb{N}. \end{array}\right. \tag{1.12}$$

Recently Phuengrattana and Suantai introduced an SP iteration method [16] by

$$\left\{\begin{array}{l} x_0 \in E, \\ x_{n+1} = (1-\alpha_n)\,y_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)\,z_n + \beta_n T z_n, \\ z_n = (1-\gamma_n)\,x_n + \gamma_n T x_n,\ \ n \in \mathbb{N}. \end{array}\right. \tag{1.13}$$

The following iteration scheme is first employed in [11] as a special case of Jungck multistep-SP iteration [12]: For arbitrary fixed order $k \geq 2$,

$$\left\{\begin{array}{l} x_0 \in E, \\ x_{n+1} = (1-\alpha_n)\,y_n^1 + \alpha_n T y_n^1, \\ y_n^1 = \left(1-\beta_n^1\right) y_n^2 + \beta_n^1 T y_n^2, \\ y_n^2 = \left(1-\beta_n^2\right) y_n^3 + \beta_n^2 T y_n^3, \\ \quad \cdots \\ y_n^{k-2} = \left(1-\beta_n^{k-2}\right) y_n^{k-1} + \beta_n^{k-2} T y_n^{k-1}, \\ y_n^{k-1} = \left(1-\beta_n^{k-1}\right) x_n + \beta_n^{k-1} T x_n,\ \ n \in \mathbb{N}. \end{array}\right. \tag{1.14}$$

or, in short,

$$
\left\{
\begin{array}{rl}
x_0 \in E, & \\
x_{n+1} = (1-\alpha_n)\, y_n^1 + \alpha_n T y_n^1, & \\
y_n^i = \left(1-\beta_n^i\right) y_n^{i+1} + \beta_n^i T y_n^{i+1}, & i = \overline{1, k-2}, \\
y_n^{k-1} = \left(1-\beta_n^{k-1}\right) x_n + \beta_n^{k-1} T x_n, & n \in \mathbb{N}.
\end{array}
\right. \tag{1.15}
$$

where, $\{\alpha_n\}_{n=0}^{\infty} \subset [0,1)$ is a sequence of real numbers such that,

$$
\sum_{n=0}^{\infty} \alpha_n = \infty \tag{1.16}
$$

and

$$
\left\{\beta_n^i\right\}_{n=0}^{\infty} \subset [0,1)\,, \quad i = \overline{1, k-1}. \tag{1.17}
$$

Remark 1. *If $\gamma_n = 0$, then SP iteration* (1.13) *reduces to iterative scheme* (1.12)*. By taking $k = 3$ and $k = 2$ in* (1.15) *we obtain the iterations* (1.13) *and* (1.12)*, respectively.*

The following Lemma will be useful to prove the main results of this work and is important by itself.

Lemma 1. [31] *Let $\{a_n\}_{n=0}^{\infty}$ be a nonnegative sequence which satisfies the following inequality*

$$
a_{n+1} \leq (1-\mu_n)\, a_n + \rho_n, \tag{1.18}
$$

where $\mu_n \in (0,1)$, for all $n \geq n_0$, $\sum\limits_{n=0}^{\infty} \mu_n = \infty$, and $\rho_n = o(\mu_n)$. Then $\lim_{n\to\infty} a_n = 0$.

2. Main Results

Theorem 1. *Let $T : E \to E$ be an operator satisfying condition* (1.4) *with $F_T \neq \emptyset$. If $x_0 = u_0 \in E$ and $\alpha_n \geq A > 0, \forall n \in \mathbb{N}$, then the following are equivalent:*

1. *The Mann iteration* (1.6) *converges to $p \in F_T$,*
2. *The new multistep iteration* (1.15) *converges to $p \in F_T$.*

Proof. Firstly, we start to prove the implication (1) $\Rightarrow$ (2): Suppose that the Mann iteration (1.6) converges to p. Using (1.6), (1.15), and (1.4) we have the following estimates:

$$
\begin{aligned}
\|u_{n+1} - x_{n+1}\| &= \left\|(1-\alpha_n)\left(u_n - y_n^1\right) + \alpha_n\left(Tu_n - Ty_n^1\right)\right\| \\
&\leq (1-\alpha_n)\left\|u_n - y_n^1\right\| + \alpha_n\left\|Tu_n - Ty_n^1\right\| \\
&\leq (1-\alpha_n)\left\|u_n - y_n^1\right\| + \alpha_n\left\{\delta\left\|u_n - y_n^1\right\| + \varphi\left(\|u_n - Tu_n\|\right)\right\} \\
&= \left[1 - \alpha_n(1-\delta)\right]\left\|u_n - y_n^1\right\| + \alpha_n \varphi\left(\|u_n - Tu_n\|\right),
\end{aligned} \tag{2.1}
$$

$$\begin{aligned}
\left\|u_n-y_n^1\right\| &= \left\|u_n-\left(1-\beta_n^1\right)y_n^2-\beta_n^1 Ty_n^2\right\| \\
&= \left\|u_n-\beta_n^1 u_n+\beta_n^1 u_n-\left(1-\beta_n^1\right)y_n^2-\beta_n^1 Ty_n^2\right\| \\
&= \left\|\left(1-\beta_n^1\right)\left(u_n-y_n^2\right)+\beta_n^1\left(u_n-Ty_n^2\right)\right\| \\
&\leq \left(1-\beta_n^1\right)\left\|u_n-y_n^2\right\|+\beta_n^1\left\|u_n-Ty_n^2\right\| \\
&= \left(1-\beta_n^1\right)\left\|u_n-y_n^2\right\|+\beta_n^1\left\|u_n-Tu_n+Tu_n-Ty_n^2\right\| \\
&\leq \left(1-\beta_n^1\right)\left\|u_n-y_n^2\right\|+\beta_n^1\left\|Tu_n-Ty_n^2\right\|+\beta_n^1\left\|u_n-Tu_n\right\| \\
&\leq \left(1-\beta_n^1\right)\left\|u_n-y_n^2\right\|+\beta_n^1\delta\left\|u_n-y_n^2\right\|+\beta_n^1\varphi\left(\left\|u_n-Tu_n\right\|\right) \\
&\quad +\beta_n^1\left\|u_n-Tu_n\right\| \\
&= \left[1-\beta_n^1(1-\delta)\right]\left\|u_n-y_n^2\right\|+\beta_n^1\left\{\left\|u_n-Tu_n\right\|+\varphi\left(\left\|u_n-Tu_n\right\|\right)\right\}, \tag{2.2}
\end{aligned}$$

$$\begin{aligned}
\left\|u_n-y_n^2\right\| &= \left\|\left(1-\beta_n^2\right)\left(u_n-y_n^3\right)+\beta_n^2\left(u_n-Ty_n^3\right)\right\| \\
&\leq \left(1-\beta_n^2\right)\left\|u_n-y_n^3\right\|+\beta_n^2\left\|u_n-Ty_n^3\right\| \\
&\leq \left(1-\beta_n^2\right)\left\|u_n-y_n^3\right\|+\beta_n^2\left\|Tu_n-Ty_n^3\right\|+\beta_n^2\left\|u_n-Tu_n\right\| \\
&\leq \left(1-\beta_n^2\right)\left\|u_n-y_n^3\right\|+\beta_n^2\delta\left\|u_n-y_n^3\right\|+\beta_n^2\varphi\left(\left\|u_n-Tu_n\right\|\right) \\
&\quad +\beta_n^2\left\|u_n-Tu_n\right\| \\
&= \left[1-\beta_n^2(1-\delta)\right]\left\|u_n-y_n^3\right\|+\beta_n^2\left\{\left\|u_n-Tu_n\right\|+\varphi\left(\left\|u_n-Tu_n\right\|\right)\right\}, \tag{2.3}
\end{aligned}$$

$$\begin{aligned}
\left\|u_n-y_n^3\right\| &= \left\|\left(1-\beta_n^3\right)\left(u_n-y_n^4\right)+\beta_n^3\left(u_n-Ty_n^4\right)\right\| \\
&\leq \left(1-\beta_n^3\right)\left\|u_n-y_n^4\right\|+\beta_n^3\left\|u_n-Ty_n^4\right\| \\
&\leq \left(1-\beta_n^3\right)\left\|u_n-y_n^4\right\|+\beta_n^3\left\|Tu_n-Ty_n^4\right\|+\beta_n^3\left\|u_n-Tu_n\right\| \\
&\leq \left(1-\beta_n^3\right)\left\|u_n-y_n^4\right\|+\beta_n^3\delta\left\|u_n-y_n^4\right\|+\beta_n^3\varphi\left(\left\|u_n-Tu_n\right\|\right) \\
&\quad +\beta_n^3\left\|u_n-Tu_n\right\| \\
&= \left[1-\beta_n^3(1-\delta)\right]\left\|u_n-y_n^4\right\|+\beta_n^3\left\{\left\|u_n-Tu_n\right\|+\varphi\left(\left\|u_n-Tu_n\right\|\right)\right\}. \tag{2.4}
\end{aligned}$$

By combinig (2.1), (2.2), (2.3), and (2.4) we obtain

$$\begin{aligned}
\left\|u_{n+1}-x_{n+1}\right\| \leq\ & \left[1-\alpha_n(1-\delta)\right]\left[1-\beta_n^1(1-\delta)\right]\left[1-\beta_n^2(1-\delta)\right] \\
& \left[1-\beta_n^3(1-\delta)\right]\left\|u_n-y_n^4\right\| \\
& +\left[1-\alpha_n(1-\delta)\right]\left\{\left[1-\beta_n^1(1-\delta)\right]\left[1-\beta_n^2(1-\delta)\right]\beta_n^3\right. \\
& \left.+\left[1-\beta_n^1(1-\delta)\right]\beta_n^2+\beta_n^1\right\}\left\{\left\|u_n-Tu_n\right\|+\varphi\left(\left\|u_n-Tu_n\right\|\right)\right\} \\
& +\alpha_n\varphi\left(\left\|u_n-Tu_n\right\|\right) \tag{2.5}
\end{aligned}$$

Continuing the above process we have

$$\begin{aligned}
\left\|u_{n+1}-x_{n+1}\right\| \leq\ & \left[1-\alpha_n(1-\delta)\right]\left[1-\beta_n^1(1-\delta)\right]\cdots\left[1-\beta_n^{k-2}(1-\delta)\right]\left\|u_n-y_n^{k-1}\right\| \\
& +\left[1-\alpha_n(1-\delta)\right]\left\{\left[1-\beta_n^1(1-\delta)\right]\cdots\left[1-\beta_n^{k-3}(1-\delta)\right]\beta_n^{k-2}\right. \\
& \left.+\cdots+\left[1-\beta_n^1(1-\delta)\right]\beta_n^2+\beta_n^1\right\}\left\{\left\|u_n-Tu_n\right\|+\varphi\left(\left\|u_n-Tu_n\right\|\right)\right\} \\
& +\alpha_n\varphi\left(\left\|u_n-Tu_n\right\|\right). \tag{2.6}
\end{aligned}$$

Again using (1.15), and (1.4) we get

$$\begin{aligned}
\left\|u_n-y_n^{k-1}\right\| &= \left\|\left(1-\beta_n^{k-1}\right)(u_n-x_n)+\beta_n^{k-1}(u_n-Tx_n)\right\| \\
&\leq \left(1-\beta_n^{k-1}\right)\|u_n-x_n\|+\beta_n^{k-1}\|u_n-Tx_n\| \\
&\leq \left(1-\beta_n^{k-1}\right)\|u_n-x_n\|+\beta_n^{k-1}\|Tu_n-Tx_n\|+\beta_n^{k-1}\|u_n-Tu_n\| \\
&\leq \left[1-\beta_n^{k-1}(1-\delta)\right]\|u_n-x_n\|+\beta_n^{k-1}\left\{\|u_n-Tu_n\|+\varphi(\|u_n-Tu_n\|)\right\}. \qquad (2.7)
\end{aligned}$$

Since $\delta\in[0,1)$ and $\{\alpha_n\}_{n=0}^{\infty},\left\{\beta_n^i\right\}_{n=0}^{\infty}\subset[0,1)$ for $i=\overline{1,k-1}$, we have

$$[1-\alpha_n(1-\delta)]\left[1-\beta_n^1(1-\delta)\right]\cdots\left[1-\beta_n^{k-1}(1-\delta)\right]\leq[1-\alpha_n(1-\delta)]. \qquad (2.8)$$

Using inequality (2.8) and the assumption $\alpha_n\geq A>0,\forall n\in\mathbb{N}$ in resultant inequality obtained by substituting (2.7) in (2.6) we get

$$\begin{aligned}
\|u_{n+1}-x_{n+1}\| \leq\ & [1-A(1-\delta)]\|u_n-x_n\| \\
&+[1-A(1-\delta)]\left\{\left[1-\beta_n^1(1-\delta)\right]\cdots\left[1-\beta_n^{k-2}(1-\delta)\right]\beta_n^{k-1}\right. \\
&\left.+\cdots+\left[1-\beta_n^1(1-\delta)\right]\beta_n^2+\beta_n^1\right\}\left\{\|u_n-Tu_n\|+\varphi(\|u_n-Tu_n\|)\right\} \\
&+\alpha_n\varphi(\|u_n-Tu_n\|). \qquad (2.9)
\end{aligned}$$

Define

$$\begin{aligned}
a_n &:= \|u_n-x_n\|, \\
\mu_n &:= A(1-\delta)\in(0,1), \\
\rho_n &:= [1-A(1-\delta)]\left\{\left[1-\beta_n^1(1-\delta)\right]\cdots\left[1-\beta_n^{k-2}(1-\delta)\right]\beta_n^{k-1}\right. \\
&\quad \left.+\cdots+\left[1-\beta_n^1(1-\delta)\right]\beta_n^2+\beta_n^1\right\}\left\{\|u_n-Tu_n\|+\varphi(\|u_n-Tu_n\|)\right\} \\
&\quad +\alpha_n\varphi(\|u_n-Tu_n\|).
\end{aligned}$$

Since $\lim_{n\to\infty}\|u_n-p\|=0$ and $Tp=p\in F_T$, it follows from (1.4) that

$$\begin{aligned}
0 &\leq \|u_n-Tu_n\| \\
&\leq \|u_n-p\|+\|Tp-Tu_n\| \\
&\leq \|u_n-p\|+\delta\|p-u_n\|+\varphi(\|p-Tp\|) \\
&= (1+\delta)\|u_n-p\|\to 0 \text{ as } n\to\infty, \qquad (2.10)
\end{aligned}$$

which implies $\lim_{n\to\infty}\|u_n-Tu_n\|=0$; namely $\rho_n=o(\mu_n)$. Hence an application of Lemma 1 to (2.10) yields $\lim_{n\to\infty}\|u_n-x_n\|=0$. Since $u_n\to p$ as $n\to\infty$ by assumption, we derive

$$\|x_n-p\|\leq\|x_n-u_n\|+\|u_n-p\| \qquad (2.11)$$

and this implies that $\lim_{n\to\infty}x_n=p$.

(2) $\Rightarrow$ (1) : Assume $x_n\to p$ as $n\to\infty$. Using (1.6), (1.15) and (1.4), we have the following estimates:

$$\begin{aligned}
\|x_{n+1}-u_{n+1}\| &= \left\|(1-\alpha_n)\left(y_n^1-u_n\right)+\alpha_n\left(Ty_n^1-Tu_n\right)\right\| \\
&\leq (1-\alpha_n)\left\|y_n^1-u_n\right\|+\alpha_n\left\|Ty_n^1-Tu_n\right\| \\
&\leq (1-\alpha_n)\left\|y_n^1-u_n\right\|+\alpha_n\left\{\delta\left\|y_n^1-u_n\right\|+\varphi\left(\left\|y_n^1-Ty_n^1\right\|\right)\right\} \\
&= [1-\alpha_n(1-\delta)]\left\|y_n^1-u_n\right\|+\alpha_n\varphi\left(\left\|y_n^1-Ty_n^1\right\|\right), \qquad (2.12)
\end{aligned}$$

$$\begin{aligned}
\left\|y_n^1-u_n\right\| &= \left\|\left(1-\beta_n^1\right)y_n^2+\beta_n^1 Ty_n^2-u_n\right\| \\
&= \left\|\left(1-\beta_n^1\right)y_n^2+\beta_n^1 Ty_n^2-u_n\left(1-\beta_n^1+\beta_n^1\right)\right\| \\
&= \left\|\left(1-\beta_n^1\right)\left(y_n^2-u_n\right)+\beta_n^1\left(Ty_n^2-u_n\right)\right\| \\
&\leq \left(1-\beta_n^1\right)\left\|y_n^2-u_n\right\|+\beta_n^1\left\|Ty_n^2-u_n\right\| \\
&\leq \left(1-\beta_n^1\right)\left\|y_n^2-u_n\right\|+\beta_n^1\left\|Ty_n^2-y_n^2+y_n^2-u_n\right\| \\
&\leq \left(1-\beta_n^1\right)\left\|y_n^2-u_n\right\|+\beta_n^1\left\|y_n^2-u_n\right\|+\beta_n^1\left\|Ty_n^2-y_n^2\right\| \\
&= \left\|y_n^2-u_n\right\|+\beta_n^1\left\|Ty_n^2-y_n^2\right\|, \qquad (2.13)
\end{aligned}$$

$$\begin{aligned}
\left\|y_n^2-u_n\right\| &= \left\|\left(1-\beta_n^2\right)y_n^3+\beta_n^2 Ty_n^3-u_n\right\| \\
&= \left\|\left(1-\beta_n^2\right)\left(y_n^3-u_n\right)+\beta_n^2\left(Ty_n^3-u_n\right)\right\| \\
&\leq \left(1-\beta_n^2\right)\left\|y_n^3-u_n\right\|+\beta_n^2\left\|Ty_n^3-u_n\right\| \\
&\leq \left(1-\beta_n^2\right)\left\|y_n^3-u_n\right\|+\beta_n^2\left\|y_n^3-u_n\right\|+\beta_n^2\left\|Ty_n^3-y_n^3\right\| \\
&= \left\|y_n^3-u_n\right\|+\beta_n^2\left\|Ty_n^3-y_n^3\right\|. \qquad (2.14)
\end{aligned}$$

By combining (2.12), (2.13), and (2.14) we obtain

$$\begin{aligned}
\left\|x_{n+1}-u_{n+1}\right\| &\leq \left[1-\alpha_n\left(1-\delta\right)\right]\left\|y_n^3-u_n\right\|+\left[1-\alpha_n\left(1-\delta\right)\right]\beta_n^2\left\|Ty_n^3-y_n^3\right\| \\
&\quad +\left[1-\alpha_n\left(1-\delta\right)\right]\beta_n^1\left\|Ty_n^2-y_n^2\right\|+\alpha_n\varphi\left(\left\|y_n^1-Ty_n^1\right\|\right) \qquad (2.15)
\end{aligned}$$

Continuing in a similar way, we have

$$\begin{aligned}
\left\|x_{n+1}-u_{n+1}\right\| &\leq \left[1-\alpha_n\left(1-\delta\right)\right]\left\|y_n^{k-1}-u_n\right\| \\
&\quad +\left[1-\alpha_n\left(1-\delta\right)\right]\beta_n^{k-2}\left\|Ty_n^{k-1}-y_n^{k-1}\right\| \\
&\quad +\cdots+\left[1-\alpha_n\left(1-\delta\right)\right]\beta_n^1\left\|Ty_n^2-y_n^2\right\|+\alpha_n\varphi\left(\left\|y_n^1-Ty_n^1\right\|\right) \qquad (2.16)
\end{aligned}$$

Using now (1.15) we have

$$\begin{aligned}
\left\|y_n^{k-1}-u_n\right\| &= \left\|\left(1-\beta_n^{k-1}\right)x_n+\beta_n^{k-1}Tx_n-u_n\right\| \\
&\leq \left(1-\beta_n^{k-1}\right)\left\|x_n-u_n\right\|+\beta_n^{k-1}\left\|Tx_n-u_n\right\| \\
&\leq \left(1-\beta_n^{k-1}\right)\left\|x_n-u_n\right\|+\beta_n^{k-1}\left\|x_n-u_n\right\|+\beta_n^{k-1}\left\|Tx_n-x_n\right\| \\
&\leq \left\|x_n-u_n\right\|+\beta_n^{k-1}\left\|Tx_n-x_n\right\|. \qquad (2.17)
\end{aligned}$$

Substituting (2.17) in (2.16) and utilizing the assumption $\alpha_n \geq A > 0, \forall n \in \mathbb{N}$ we get

$$\begin{aligned}
\left\|x_{n+1}-u_{n+1}\right\| &\leq \left[1-A\left(1-\delta\right)\right]\left\|x_n-u_n\right\| \\
&\quad +\left[1-A\left(1-\delta\right)\right]\left\{\beta_n^{k-1}\left\|Tx_n-x_n\right\|+\beta_n^{k-2}\left\|Ty_n^{k-1}-y_n^{k-1}\right\|\right. \\
&\quad \left.+\cdots+\beta_n^1\left\|Ty_n^2-y_n^2\right\|\right\}+\alpha_n\varphi\left(\left\|y_n^1-Ty_n^1\right\|\right). \qquad (2.18)
\end{aligned}$$

Now define

$$\begin{aligned}
a_n &: = \left\|u_n-x_n\right\|, \\
\mu_n &: = A\left(1-\delta\right)\in(0,1), \\
\rho_n &: = +\left[1-A\left(1-\delta\right)\right]\left\{\beta_n^{k-1}\left\|Tx_n-x_n\right\|+\beta_n^{k-2}\left\|Ty_n^{k-1}-y_n^{k-1}\right\|\right. \\
&\quad \left.+\cdots+\beta_n^1\left\|Ty_n^2-y_n^2\right\|\right\}+\alpha_n\varphi\left(\left\|y_n^1-Ty_n^1\right\|\right).
\end{aligned}$$

Since $\lim_{n\to\infty}\|x_n-p\|=0$ and $Tp=p\in F_T$, it follows from (1.4) that

$$\begin{aligned}
0 &\leq \|x_n-Tx_n\| \\
&\leq \|x_n-p\|+\|Tp-Tx_n\| \\
&\leq \|x_n-p\|+\delta\|p-x_n\|+\varphi(\|p-Tp\|) \\
&= (1+\delta)\|x_n-p\|\to 0 \text{ as } n\to\infty. \qquad (2.19)
\end{aligned}$$

Utilizing (1.4), (1.15), and (1.17), we have

$$\begin{aligned}
0 &\leq \left\|y_n^1-Ty_n^1\right\|=\left\|y_n^1-p+p-Ty_n^1\right\| \\
&\leq \left\|y_n^1-p\right\|+\left\|Tp-Ty_n^1\right\| \\
&\leq \left\|y_n^1-p\right\|+\delta\left\|p-y_n^1\right\|+\varphi(\|p-Tp\|) \\
&= (1+\delta)\left\|y_n^1-p\right\| \\
&= (1+\delta)\left\|\left(1-\beta_n^1\right)y_n^2+\beta_n^1Ty_n^2-p\left(1-\beta_n^1+\beta_n^1\right)\right\| \\
&\leq (1+\delta)\left\{\left(1-\beta_n^1\right)\left\|y_n^2-p\right\|+\beta_n^1\left\|Ty_n^2-Tp\right\|\right\} \\
&\leq (1+\delta)\left\{\left(1-\beta_n^1\right)\left\|y_n^2-q\right\|+\beta_n^1\delta\left\|y_n^2-p\right\|\right\} \\
&= (1+\delta)\left[1-\beta_n^1(1-\delta)\right]\left\|y_n^2-p\right\| \\
&= (1+\delta)\left[1-\beta_n^1(1-\delta)\right]\left\|\left(1-\beta_n^2\right)y_n^3+\beta_n^2Ty_n^3-p\left(1-\beta_n^2+\beta_n^2\right)\right\| \\
&\leq (1+\delta)\left[1-\beta_n^1(1-\delta)\right]\left\{\left(1-\beta_n^2\right)\left\|y_n^3-q\right\|+\beta_n^2\left\|Ty_n^3-Tp\right\|\right\} \\
&\leq (1+\delta)\left[1-\beta_n^1(1-\delta)\right]\left[1-\beta_n^2(1-\delta)\right]\left\|y_n^3-p\right\| \\
&\cdots \\
&\leq (1+\delta)\left[1-\beta_n^1(1-\delta)\right]\cdots\left[1-\beta_n^{k-2}(1-\delta)\right]\left\|y_n^{k-1}-p\right\| \\
&\leq (1+\delta)\left[1-\beta_n^1(1-\delta)\right]\cdots\left[1-\beta_n^{k-1}(1-\delta)\right]\|x_n-p\| \\
&\leq (1+\delta)\|x_n-p\|\to 0 \text{ as } n\to\infty. \qquad (2.20)
\end{aligned}$$

It is easy to see from (2.20) that this result is also valid for $\left\|Ty_n^2-y_n^2\right\|,\ldots,\left\|Ty_n^{k-1}-y_n^{k-1}\right\|$.

Since φ is continuous, we have

$$\begin{aligned}
\lim_{n\to\infty}\|x_n-Tx_n\| &= \lim_{n\to\infty}\varphi\left(\left\|y_n^1-Ty_n^1\right\|\right) \\
&= \lim_{n\to\infty}\left\|y_n^2-Ty_n^2\right\|=\cdots=\lim_{n\to\infty}\left\|y_n^{k-1}-Ty_n^{k-1}\right\|=0, \qquad (2.21)
\end{aligned}$$

that is $\rho_n=o(\mu_n)$. Hence an application of Lemma 1 to (2.18) lead to $\lim_{n\to\infty}\|x_n-u_n\|=0$. Since $x_n\to p$ as $n\to\infty$ by assumption, we derive

$$\|u_n-p\|\leq\|u_n-x_n\|+\|x_n-p\| \qquad (2.22)$$

and this implies that $\lim_{n\to\infty}u_n=p$. □

Theorem 2. *Let $T:E\to E$ be an operator satisfying condition* (1.4) *with $F_T\neq\emptyset$. If $x_0=u_0\in E$ and $\alpha_n\geq A>0,\forall n\in\mathbb{N}$, then the following are equivalent:*

1. *The Mann iteration* (1.6) *converges to $p\in F_T$,*
2. *The S-iteration* (1.11) *converges to $p\in F_T$.*

Proof. Firstly, we start to prove the implication (1) ⇒ (2): Suppose that the Mann iteration (1.6) converges to p. Using (1.4), (1.6), and (1.11) we have the following

estimates:

$$\begin{aligned}\|u_{n+1}-x_{n+1}\| &= \|(1-\alpha_n)(u_n-Tx_n)+\alpha_n(Tu_n-Ty_n)\| \\ &\le (1-\alpha_n)\|u_n-Tx_n\|+\alpha_n\|Tu_n-Ty_n\| \\ &\le (1-\alpha_n)\|u_n-Tx_n\|+\alpha_n\delta\|u_n-y_n\|+\alpha_n\varphi(\|u_n-Tu_n\|), \quad (2.23)\end{aligned}$$

$$\begin{aligned}\|u_n-y_n\| &= \|u_n-(1-\beta_n)x_n-\beta_nTx_n\| \\ &= \|u_n-\beta_nu_n+\beta_nu_n-(1-\beta_n)x_n-\beta_nTx_n\| \\ &\le (1-\beta_n)\|u_n-x_n\|+\beta_n\|u_n-Tx_n\|, \quad (2.24)\end{aligned}$$

$$\begin{aligned}\|u_n-Tx_n\| &= \|u_n-Tu_n+Tu_n-Tx_n\| \\ &\le \|u_n-Tu_n\|+\|Tu_n-Tx_n\| \\ &\le \|u_n-Tu_n\|+\delta\|u_n-x_n\|+\varphi(\|u_n-Tu_n\|). \quad (2.25)\end{aligned}$$

By combining (2.23),(2.24), and (2.25) we obtain

$$\begin{aligned}\|u_{n+1}-x_{n+1}\| \le\ & \{(1-\alpha_n)\delta+\alpha_n\delta[1-\beta_n(1-\delta)]\}\|u_n-x_n\| \\ &+[1-\alpha_n+\alpha_n\beta_n\delta]\|u_n-Tu_n\| \\ &+[1+\alpha_n\beta_n\delta]\varphi(\|u_n-Tu_n\|). \quad (2.26)\end{aligned}$$

Since $\delta,\alpha_n,\beta_n\in[0,1)$ for all $n\in\mathbb{N}$,

$$(1-\alpha_n)\delta<1-\alpha_n,\ 1-\beta_n(1-\delta)<1. \quad (2.27)$$

Using (2.27) and the assumption $\alpha_n\ge A>0,\forall n\in\mathbb{N}$ in (2.26) we derive

$$\begin{aligned}\|u_{n+1}-x_{n+1}\| \le\ & [1-A(1-\delta)]\|u_n-x_n\| \\ &+[1-A(1-\delta)]\|u_n-Tu_n\| \\ &+[1+\alpha_n\beta_n\delta]\varphi(\|u_n-Tu_n\|). \quad (2.28)\end{aligned}$$

Define

$$\begin{aligned}a_n &: =\|u_n-x_n\|, \\ \mu_n &: =A(1-\delta)\in(0,1), \\ \rho_n &: =[1-A(1-\delta)]\|u_n-Tu_n\|+[1+\alpha_n\beta_n\delta]\varphi(\|u_n-Tu_n\|).\end{aligned}$$

Since $\lim_{n\to\infty}\|u_n-p\|=0$, $\lim_{n\to\infty}\|u_n-Tu_n\|=0$ as in the proof of Theorem1. It therefore follows from the same argument that employed in the proof of Theorem 1 that $\lim_{n\to\infty}x_n=p$.

We will prove now that if the S-iteration converges, then the Mann iteration does too.

Using (1.4), (1.6), and (1.11) we have

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| &= \|(1-\alpha_n)(Tx_n-u_n)+\alpha_n(Ty_n-Tu_n)\| \\ &\le (1-\alpha_n)\|Tx_n-u_n\|+\alpha_n\|Ty_n-Tu_n\| \\ &\le (1-\alpha_n)\|Tx_n-u_n\|+\alpha_n\delta\|y_n-u_n\|+\alpha_n\varphi(\|y_n-Ty_n\|). \quad (2.29)\end{aligned}$$

We now have the following estimates

$$\begin{aligned}\|y_n-u_n\| &= \|(1-\beta_n)x_n+\beta_nTx_n-u_n\| \\ &= \|(1-\beta_n)x_n+\beta_nTx_n-u_n-\beta_nu_n+\beta_nu_n\| \\ &\le (1-\beta_n)\|x_n-u_n\|+\beta_n\|Tx_n-u_n\|, \quad (2.30)\end{aligned}$$

$$\begin{aligned} \|Tx_n - u_n\| &= \|Tx_n - x_n + x_n - u_n\| \\ &\leq \|Tx_n - x_n\| + \|x_n - u_n\| . \end{aligned} \tag{2.31}$$

Relations (2.29),(2.30), and (2.31) lead to

$$\begin{aligned} \|x_{n+1} - u_{n+1}\| &\leq [1 - \alpha_n (1 - \delta)] \|x_n - u_n\| \\ &\quad + [1 - \alpha_n + \alpha_n \beta_n \delta] \|Tx_n - x_n\| + \alpha_n \varphi (\|y_n - Ty_n\|) . \end{aligned} \tag{2.32}$$

Since $\beta_n \in [0, 1)$ for all $n \in \mathbb{N}$,

$$\alpha_n \beta_n \delta < \alpha_n \delta. \tag{2.33}$$

Utilizing inequality (2.33) and the assumption $\alpha_n \geq A > 0, \forall n \in \mathbb{N}$ in (2.32) we get

$$\begin{aligned} \|u_{n+1} - x_{n+1}\| &\leq [1 - A (1 - \delta)] \|x_n - u_n\| \\ &\quad + [1 - A (1 - \delta)] \|Tx_n - x_n\| + \alpha_n \varphi (\|y_n - Ty_n\|) . \end{aligned} \tag{2.34}$$

Now define

$$\begin{aligned} a_n &: = \|x_n - u_n\| , \\ \mu_n &: = A (1 - \delta) \in (0, 1) , \\ \rho_n &: = [1 - A (1 - \delta)] \|Tx_n - x_n\| + \alpha_n \varphi (\|y_n - Ty_n\|) . \end{aligned}$$

Since $\lim_{n\to\infty} \|x_n - p\| = 0$, $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$ as in the proof of Theorem1.

Now we have

$$\begin{aligned} 0 &\leq \|y_n - Ty_n\| \\ &\leq \|y_n - p\| + \|Tp - Ty_n\| \\ &\leq \|y_n - p\| + \delta \|p - y_n\| + \varphi (\|p - Tp\|) \\ &= (1 + \delta) \|y_n - p\| \\ &\leq (1 + \delta) (1 - \beta_n) \|x_n - p\| + (1 + \delta) \beta_n \|Tx_n - Tp\| \\ &\leq (1 + \delta) [1 - \beta_n + \beta_n] \|x_n - p\| + (1 + \delta) \beta_n \varphi (\|p - Tp\|) \\ &= (1 + \delta) \|x_n - p\| \to 0 \text{ as } n \to \infty, \end{aligned} \tag{2.35}$$

that is, $\lim_{n\to\infty} \|y_n - Ty_n\| = 0$, threfore using the same argument as in the proof of Theorem 1, it can be shown that $\lim_{n\to\infty} u_n = p$. □

As shown by Şoltuz and Grosan ([24], Theorem 3.1), in a real Banach space X, the Ishikawa iteration $\{x_n\}_{n=0}^{\infty}$ given by (1.7) converges to the fixed point of T, where $T : E \to E$ is a mapping satisfying condition (1.4).

In 2007, Şoltuz ([25], Corollary 2) proved that Krasnoselskij (1.7), Mann (1.6), Ishikawa (1.8), Noor (1.9) and multistep (1.10) iterations are equivalent for quasi-contractive operators in a normed space setting.

In 2011, Chugh and Kumar ([19], Corollary 3.2) proved that Picard (1.5), Mann (1.6), Ishikawa (1.8), new two step (1.12), Noor (1.9) and SP (1.13) iterations are equivalent for quasi-contractive operators in a Banach space setting.

From the argument used in the proofs of ([24], Theorem 3.1), ([25], Corollary 2) and ([19], Corollary 3.2) we can easily get the following corollary:

Corollary 1. *$T : E \to E$ be an operator satisfying condition* (1.4) *with $F_T \neq \emptyset$. If the initial point is the same for all iterations, $\alpha_n \geq A > 0$, $\forall n \in \mathbb{N}$, then the following are equivalent:*

(1) *The Picard iteration* (1.5) *converges to $p \in F_T$;*

(2) *The Krasnoselskij iteration* (1.7) *converges to* $p \in F_T$.
(3) *The Mann iteration* (1.6) *converges to* $p \in F_T$;
(4) *The Ishikawa iteration* (1.8) *converges to* $p \in F_T$;
(5) *The new two step iteration* (1.12) *converges to* $p \in F_T$;
(6) *The Noor iteration* (1.9) *converges to* $p \in F_T$;
(7) *The SP iteration* (1.13) *converges to* $p \in F_T$;
(8) *The Multistep iteration* (1.10) *converges to* $p \in F_T$;

Together with Theorem 1 and Theorem 2,Corollary 1 leads to the following corollary:

Corollary 2. $T : E \to E$ *be an operator satisfying condition* (1.4) *with* $F_T \neq \emptyset$. *If the initial point is the same for all iterations,* $\alpha_n \geq A > 0$, $\forall n \in \mathbb{N}$, *then the following are equivalent:*

(1) *The Picard iteration* (1.5) *converges to* $p \in F_T$;
(2) *The Krasnoselskij iteration* (1.7) *converges to* $p \in F_T$.
(3) *The Mann iteration* (1.6) *converges to* $p \in F_T$;
(4) *The Ishikawa iteration* (1.8) *converges to* $p \in F_T$;
(5) *The new two step iteration* (1.12) *converges to* $p \in F_T$;
(6) *The Noor iteration* (1.9) *converges to* $p \in F_T$;
(7) *The SP iteration* (1.13) *converges to* $p \in F_T$;
(8) *The Multistep iteration* (1.10) *converges to* $p \in F_T$;
(9) *The new multistep iteration* (1.14) (*or* (1.15)) *converges to* $p \in F_T$;
(10) *The S-iteration* (1.11) *converges to* $p \in F_T$.

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